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THE MODIFIED HOMOTOPY PERTURBATION METHOD FOR THE APPROXIMATE SOLUTION OF NONLINEAR OSCILLATORS

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Abstract. The paper deals with the modified homotopy perturbation method (MHPM) and solution of some examples of nonlinear oscillators. The modification is done by adding a term to the linear operator based on the equation and boundary conditions. In modified form, only one iteration yields highly accurate solutions. The proposed new modification is compared with the classical homotopy perturbation method.

Keywords: Modified homotopy perturbation method, auxiliary parameter, strongly nonlinear oscillator, approximate solutions.

AMS Subject Classification: 34A25, 34C15, 34E15, 34K2.

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1 Introduction

Most phenomena in the world are nonlinear and modeled using oscillating systems. These systems are very important in electronics, mechanics, dynamics, nanotechnology and also modelling of global warming, epidemic diseases, sociology, chemical reactions, biology and ecology. Most of the phenomena occurring in these fields are modeled as nonlinear differential equations. It is difficult to solve nonlinear problems accurately. For this reason, a wide variety of approximate methods have been developed to solve these problems.

The classical homotopy perturbation method (HPM) is proposed first by Ji Huan He, see (He, 1999, 2003), for solving linear and nonlinear problems. The method is developed using the concept of homotopy in topology, which does not require a small parameter in an equation. This property is an important advantage in many nonlinear problems. The traditional perturbation methods are well-known with their shortcomings, hence they are not applicable for strongly nonlinear equations. As a result of many years of research, new techniques have been introduced to the literature to overcome the limitation, such as the homotopy perturbation method (He, 1999, 2003, 2004; He & El-Dib, 2021a; He at al., 2021a; Kapoor, 2020; Shou, 2009).

Some effective modifications have been developed to facilitate computations and rapid convergence of the series solution, for example, the modified homotopy perturbation method (MHPM) (Anjum & He, 2020; Anjum at al., 2021; He & El-Dib, 2021b; Pasha at al., 2019), He-Laplace method (He at al., 2021b), the frequency-amplitude formulation (He, 2021a), the gamma function method (He, 2021b), generalized Taylor series (Odibat, 2020), an analytical approach (Pankratov, 2020), the modified Lindstedt-Poincare method (Faydaoglu & Ozis, 2021; Sharif at al., 2020), the fractal semi-inverse transform method (Wang, 2021), computation of the regularized Green's function and the decomposition method for boundary value problem (Faydaoglu

& Yakhno, 2021; Yucel & Mukhtarov, 2018). In this paper is suggested MHPM for solving nonlinear differential equations governing strongly nonlinear oscillators, and compared with the traditional one.

In Section 2, the basic meaning of MHPM is explained. In Section 3, It is applied the modification of HPM to two examples with small parameters. Numerical calculations have been made with MAPLE software. The obtained solutions has been presented in graphs.

2 The main idea of the modification of HPM

The MHPM used to solve the nonlinear oscillator is similar to the classical HPM (Anjum & He, 2020; Anjum at al., 2021; He & El-Dib, 2021b; Pasha at al., 2019). A linear term such as $f(\alpha)$ is added and subtracted to the governing equation to make the modification. In this case, the governing equation does not change, but only one of the additional terms is chosen for using in the linear operator. This term added to the linear operator prevents the solution from divergence. A fast convergent and very accurate solution is obtained. The only difference between modified of HPM and classic HPM is the linear operator. Other procedures are the same in both cases. In the modified form, linear operator plays the dominant role in solution.

3 Applications of proposed MHPM methodology

In this section, we consider the following two nonlinear oscillators. We demonstrate the effectiveness of the proposed modification and compare with the classical HPM.

Example 1. Let us consider the nonlinear oscillator with initial conditions (He, 2004)

$$v'' + v + \lambda v|v| = 0, \quad v(0) = \alpha, \quad v'(0) = 0, \tag{1}$$

where $\lambda v|v|$ is the discontinuous function. To introduce the proposed MHPM, $f(\alpha)$ is defined taking into account α from the initial condition and for the periodic solution of Eq. (1), this equation is rewritten as:

$$v'' + f(\alpha)v - f(\alpha)v + 1.v + \lambda v|v| = 0. \tag{2}$$

Now, only one of the additional terms for the linear operator is taken

$$L(v) \equiv v'' + f(\alpha)v + 1.v = 0. \tag{3}$$

If we consider the solution and expansion the constant unity as

$$v = \sum_{j=0}^{k} \gamma_j e^{c_j t}, \quad 1 = \Omega^2 + \sum_{j=1}^{k} \gamma_j \Omega_j, \tag{4}$$

and substitute to the characteristic equation of governing equation, we get

$$c_j^2 (\textstyle\sum_{j=0}^{k} \gamma_j e^{c_j t}) + (\Omega^2 + \sum_{j=1}^{k} \gamma_j \Omega_j)(\sum_{j=0}^{k} \gamma_j e^{c_j t}) + \lambda(\sum_{j=0}^{k} \gamma_j e^{c_j t})|\sum_{j=0}^{k} \gamma_j e^{c_j t}| = 0. \tag{5}$$

Note that the series in Eqs. (4) may be asymptotic. Here Ω is the angular frequency of the oscillator. Ω_1 can be found in view of no secular terms in v_1. With considering $v = \sum_{j=0}^{k} \gamma_j e^{c_j t} \neq 0$, it simplifies to:

$$c_j^2 = -(\Omega^2 + \sum_{j=1}^{k} \gamma_j \Omega_j + \lambda|\sum_{j=0}^{k} \gamma_j e^{c_j t})|). \tag{6}$$

Here, we predict that the solution of (3) will serve as a leading term, v_0, in the solution series. Hence, the equation provided by v_0 should be

$$v_0'' + f(\alpha)v_0 + (\Omega^2 + \Omega_1)v_0 = 0. \tag{7}$$

The characteristic equation of this equation by using the second equation of (4) is written as

$$c_0^2 = -f(\alpha) - \Omega^2 - \Omega_1. \tag{8}$$

If we consider (6) and $v_0 = \gamma_0 e^{c_0 t} = \alpha e^{c_0 t}$ then we get

$$c_0^2 = -(\Omega^2 + \Omega_1 + \lambda|\gamma_0 e^{c_0 t}|) = -(\Omega^2 + \Omega_1 + \lambda|\alpha e^{c_0 t}|), \tag{9}$$

and from (8) and (9), since $f(\alpha)$ is an integral number we have

$$f(\alpha) = \lambda|\alpha e^{c_0 t}| = \lambda\alpha. \tag{10}$$

Substituting (10) in Eq. (2), we obtain the equality

$$v'' + (\lambda\alpha + 1)v - \lambda\alpha v + \lambda v|v| = 0. \tag{11}$$

Now, we set up the following homotopy and take the linear term as $v'' + (\lambda\alpha + 1)v$

$$v'' + (\lambda\alpha + 1)v + \rho\lambda v|v| = 0, \tag{12}$$

where $1 = \Omega^2 + \Omega_1$ (here, 1, given in (4), estimated with first order approximation and $\gamma_1 = 1$ is taken for simplicity). We use the homotopy parameter ρ to expand the solution as a power series in ρ

$$v = v_0 + \rho v_1 + \rho^2 v_2 + ... \quad . \tag{13}$$

and the constant unity in the middle term is expanded also in the series form

$$1 = \Omega^2 + \rho\Omega_1 + \rho^2\Omega_2 + ... \quad . \tag{14}$$

Substituting Eqs.(4) and the initial conditions $v(0) = \alpha$, $v'(0) = 0$ into the homotopy (12) and equating the terms with identical powers of ρ, we obtain the following set of linear differential equations

$$\rho^0 : v_0'' + (\Omega^2 + \lambda\alpha)v_0 = 0,\ v_0(0) = \alpha,\ v_0'(0) = 0, \tag{15}$$

$$\rho^1 : v_1'' + (\Omega^2 + \lambda\alpha)v_1 + (\Omega_1 + \lambda|v_0|)v_0 = 0, \quad v_1(0) = 0,\ v_1'(0) = 0. \tag{16}$$

Hence, the solution of equation (15) above is as follows

$$v_0 = \alpha cos(\sqrt{\Omega^2 + \lambda\alpha}t). \tag{17}$$

Substituting Eq. (17) into Eq. (16), we obtain a differential equation for v_1. This equation requires no secular term. Hence, we can write

$$\int_0^T cos(\Omega t)[\Omega_1 v_0 - \lambda v_0|v_0|]dt = 0, \quad T = \frac{2\pi}{\Omega}. \tag{18}$$

The first-order approximate solution is always of high accuracy and it is valid for the entire solution space (He, 2004). From Eq. (18) we can determine the angular frequency easily.

Here, if $-\frac{\pi}{2} \leq \sqrt{\Omega^2+\lambda\alpha}t \leq \frac{\pi}{2}$ then, $|cos(\sqrt{\Omega^2+\lambda\alpha}t)| = cos(\sqrt{\Omega^2+\lambda\alpha}t)$ and if $\frac{\pi}{2} \leq \sqrt{\Omega^2+\lambda\alpha}t \leq \frac{3\pi}{2}$ then, $|cos(\sqrt{\Omega^2+\lambda\alpha}t)| = -cos(\sqrt{\Omega^2+\lambda\alpha}t)$.

So we rewrite Eq. (18) in the form

$$\begin{aligned} &\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} cos(\Omega t)[\Omega_1\alpha cos(\sqrt{\Omega^2+\lambda\alpha}t) + \lambda\alpha^2 cos^2(\sqrt{\Omega^2+\lambda\alpha}t)]dt \\ &+\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} cos(\Omega t)[\Omega_1\alpha cos(\sqrt{\Omega^2+\lambda\alpha}t) - \lambda\alpha^2 cos^2(\sqrt{\Omega^2+\lambda\alpha}t)]dt = 0, \quad T = \frac{2\pi}{\Omega}. \end{aligned} \tag{19}$$

From the above equation and the second equation of (4), we can easily find that

$$\Omega_1 = -\frac{8\lambda\alpha}{3\pi}, \quad \Omega = \sqrt{1+\frac{8\lambda\alpha}{3\pi}}. \tag{20}$$

If first-order approximate solution is enough, we set $\Omega_i = 0$, $i \geq 2$ in Eq. (4). Its period reads $T = \frac{2\pi}{\sqrt{1+\frac{8\lambda\alpha}{3\pi}}}$.
In case $\lambda = 0$, its period can be written as $T = 2\pi$. We rewrite Eq. (16) in the form

$$v_1'' + (\Omega^2+\lambda\alpha)v_1 = \begin{cases} \frac{8\lambda\alpha^2}{3\pi}cos(\sqrt{\Omega^2+\lambda\alpha}t) - \lambda\alpha^2 cos^2(\sqrt{\Omega^2+\lambda\alpha}t), \\ \qquad -\frac{\pi}{2} \leq \sqrt{\Omega^2+\lambda\alpha}t \leq \frac{\pi}{2} \\ \frac{8\lambda\alpha^2}{3\pi}cos(\sqrt{\Omega^2+\lambda\alpha}t) + \lambda\alpha^2 cos^2(\sqrt{\Omega^2+\lambda\alpha}t), \\ \qquad \frac{\pi}{2} \leq \sqrt{\Omega^2+\lambda\alpha}t \leq \frac{3\pi}{2}, \end{cases} \tag{21}$$

with initial conditions $v_1(0) = 0$ and $v_1'(0) = 0$. The solution of Eq. (21) reads

$$v_1 = \begin{cases} \frac{\lambda\alpha^2}{6\pi(\Omega^2+\lambda\alpha)^{\frac{3}{2}}}[2\pi\sqrt{\Omega^2+\lambda\alpha}cos(\sqrt{\Omega^2+\lambda\alpha}t) + 8\Omega^2 tsin(\sqrt{\Omega^2+\lambda\alpha}t) \\ +8\lambda\alpha tsin(\sqrt{\Omega^2+\lambda\alpha}t) - 3\pi\sqrt{\Omega^2+\lambda\alpha} + \pi\sqrt{\Omega^2+\lambda\alpha}cos(2\sqrt{\Omega^2+\lambda\alpha}t)], \\ \qquad -\frac{\pi}{2} \leq \sqrt{\Omega^2+\lambda\alpha}t \leq \frac{\pi}{2} \\ \frac{\lambda\alpha^2}{6\pi(\Omega^2+\lambda\alpha)^{\frac{3}{2}}}[-2\pi\sqrt{\Omega^2+\lambda\alpha}cos(\sqrt{\Omega^2+\lambda\alpha}t) + 8\Omega^2 tsin(\sqrt{\Omega^2+\lambda\alpha}t) \\ +8\lambda\alpha tsin(\sqrt{\Omega^2+\lambda\alpha}t) + 3\pi\sqrt{\Omega^2+\lambda\alpha} - \pi\sqrt{\Omega^2+\lambda\alpha}cos(2\sqrt{\Omega^2+\lambda\alpha}t)], \\ \qquad \frac{\pi}{2} \leq \sqrt{\Omega^2+\lambda\alpha}t \leq \frac{3\pi}{2}, \end{cases} \tag{22}$$

Hence, we obtain the first order approximation by setting $\rho = 1$:

$$v = v_0 + v_1 = \begin{cases} \alpha cos(\sqrt{\Omega^2+\lambda\alpha}t) + \frac{\lambda\alpha^2}{6\pi(\Omega^2+\lambda\alpha)^{\frac{3}{2}}}[2\pi\sqrt{\Omega^2+\lambda\alpha}cos(\sqrt{\Omega^2+\lambda\alpha}t) \\ +8\Omega^2 tsin(\sqrt{\Omega^2+\lambda\alpha}t) + 8\lambda\alpha tsin(\sqrt{\Omega^2+\lambda\alpha}t) \\ -3\pi\sqrt{\Omega^2+\lambda\alpha} + \pi\sqrt{\Omega^2+\lambda\alpha}cos(2\sqrt{\Omega^2+\lambda\alpha}t)], \\ \qquad -\frac{\pi}{2} \leq \sqrt{\Omega^2+\lambda\alpha}t \leq \frac{\pi}{2} \\ \alpha cos(\sqrt{\Omega^2+\lambda\alpha}t) + \frac{\lambda\alpha^2}{6\pi(\Omega^2+\lambda\alpha)^{\frac{3}{2}}}[-2\pi\sqrt{\Omega^2+\lambda\alpha}cos(\sqrt{\Omega^2+\lambda\alpha}t) \\ +8\Omega^2 tsin(\sqrt{\Omega^2+\lambda\alpha}t) + 8\lambda\alpha tsin(\sqrt{\Omega^2+\lambda\alpha}t) \\ +3\pi\sqrt{\Omega^2+\lambda\alpha} - \pi\sqrt{\Omega^2+\lambda\alpha}cos(2\sqrt{\Omega^2+\lambda\alpha}t)], \\ \qquad \frac{\pi}{2} \leq \sqrt{\Omega^2+\lambda\alpha}t \leq \frac{3\pi}{2}, \end{cases} \tag{23}$$

where the angular frequency Ω is defined as Eq. (20).

The first-order approximate solution of HPM of the problem may be found as

$$v = v_0 + v_1 = \begin{cases} \alpha cos(\Omega t) + \frac{\lambda\alpha^2}{6\pi\Omega^2}[2\pi cos(\Omega t) + 8\Omega tsin(\Omega t) - 3\pi + \pi cos(2\Omega t)], \\ \qquad -\frac{\pi}{2} \leq \Omega t \leq \frac{\pi}{2} \\ \alpha cos(\Omega t) - \frac{\lambda A^2}{6\pi\Omega^2}[2\pi cos(\Omega t) - 8\Omega tsin(\Omega t) - 3\pi + \pi cos(2\Omega t)], \\ \qquad \frac{\pi}{2} \leq \Omega t \leq \frac{3\pi}{2} \end{cases} \tag{24}$$

To make the comparison more concrete, we made the numerical calculations of the solutions (23) and (24) for distinct t, α and λ values. Fig. 1, Fig. 2, Fig. 3 and Fig. 4 show the comparisons of the solutions for distinct t values and for $\alpha = 2$, $\alpha = 0.2$, $\lambda = 3$, $\lambda = 8$, $\frac{-\pi}{2} \leq \Omega t \leq \frac{\pi}{2}$, $\frac{\pi}{2} \leq \sqrt{\Omega^2+\lambda\alpha}t \leq \frac{3\pi}{2}$. It is noticible that in HPM, the difference between zero-order and 1st order solutions diverges when t gets larger. But, zero-order and 1st order solutions for MHPM are consistent even for large t values. This distinction is clearly observed on figures for HPM and MHPM. Therefore, in this example, MHPM works better comparison to HPM for least order solutions.

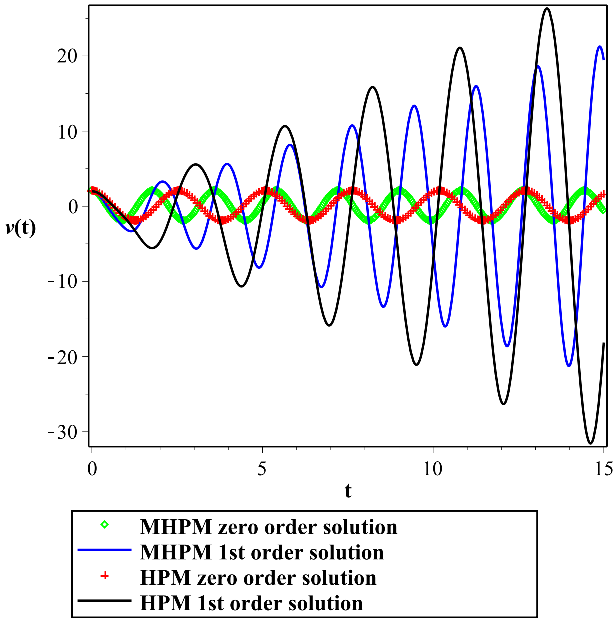

Figure 1: The comparison between zero-order and 1st order solutions for HPM and MHPM for $\alpha = 2$, $\lambda = 3$, $\frac{-\pi}{2} \leq \Omega t \leq \frac{\pi}{2}$, $\frac{-\pi}{2} \leq \sqrt{\Omega^2 + \lambda\alpha}t \leq \frac{\pi}{2}$.

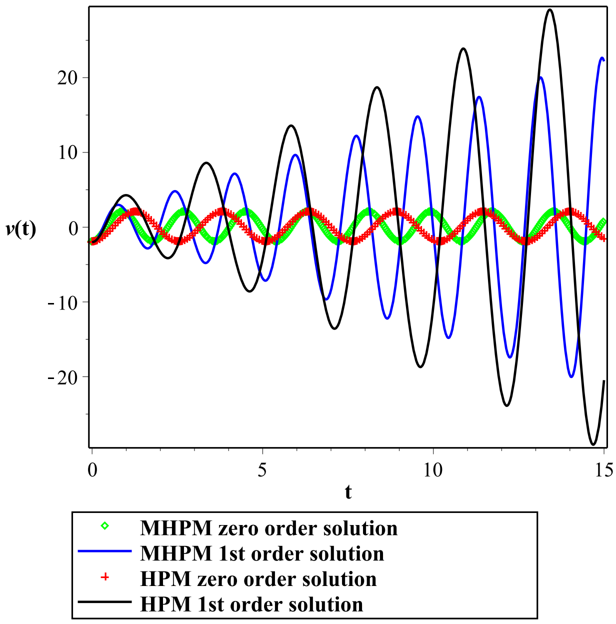

Figure 2: The comparison between zero-order and 1st order solutions for HPM and MHPM for $\alpha = 2$, $\lambda = 3$, $\frac{\pi}{2} \leq \Omega t \leq \frac{3\pi}{2}$, $\frac{\pi}{2} \leq \sqrt{\Omega^2 + \lambda\alpha}t \leq \frac{3\pi}{2}$.

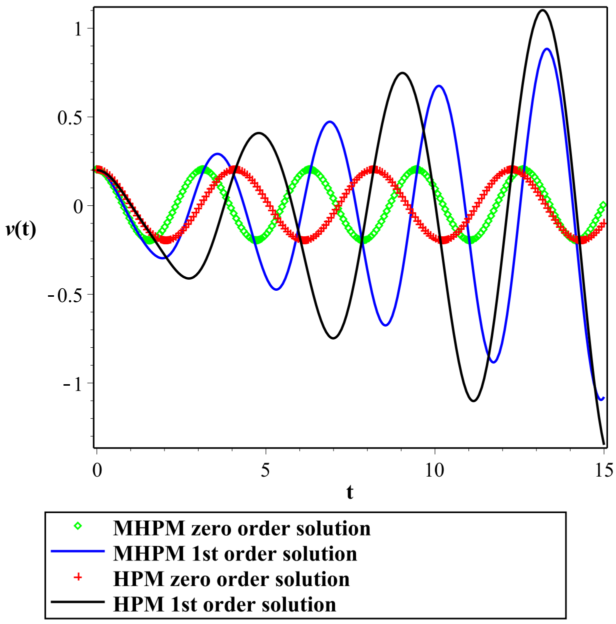

Figure 3: The comparison between zero-order and 1st order solutions for HPM and MHPM for $\alpha = 0.2$, $\lambda = 8$, $\frac{-\pi}{2} \leq \Omega t \leq \frac{\pi}{2}$, $\frac{-\pi}{2} \leq \sqrt{\Omega^2 + \lambda\alpha} t \leq \frac{\pi}{2}$.

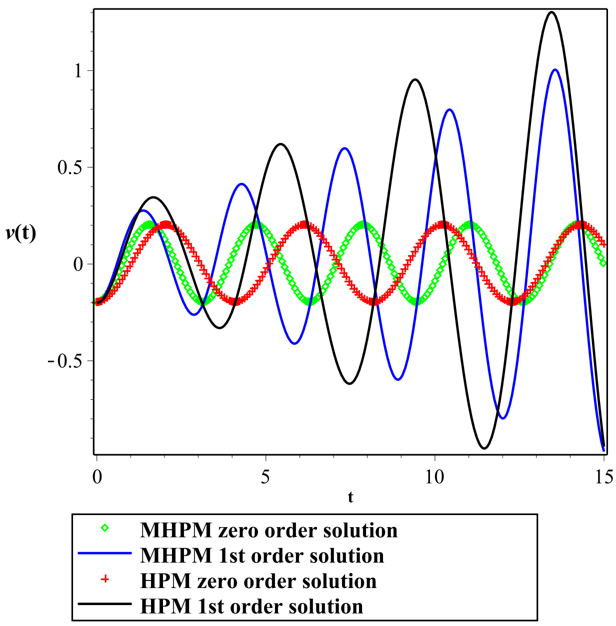

Figure 4: The comparison between zero-order and 1st order solutions for HPM and MHPM for $\alpha = 0.2$, $\lambda = 8$, $\frac{\pi}{2} \leq \Omega t \leq \frac{3\pi}{2}$, $\frac{\pi}{2} \leq \sqrt{\Omega^2 + \lambda\alpha} t \leq \frac{3\pi}{2}$.

Example 2. Duffing oscillator is an example of a periodically forced oscillator with a nonlinear elasticity, modelled as the following second order differential equation (Shou, 2009)

$$vv'' + 1 = 0, \quad v(0) = \alpha, \quad v'(0) = 0. \tag{25}$$

Now let us rewrite this equation in the form

$$v'' + v(v'')^2 = 0, \tag{26}$$

and, perform operations similar to Ex.1 to apply MHPM

$$v'' + f(\alpha)v - f(\alpha)v + 0.v + v(v'')^2 = 0. \tag{27}$$

If we consider just one of the following additional terms for linear operator, the solution and expanding the constant zero

$$L(v) \equiv v'' + f(\alpha)v + 0.v \quad , \tag{28}$$

$$v = \sum_{j=0}^{k} \gamma_j e^{c_j t}, \quad 0 = \Omega^2 + \sum_{j=1}^{k} \gamma_j \Omega_j, \tag{29}$$

and substitute to the characteristic equation of governing equation, we get

$$c_j^2 (\textstyle\sum_{j=0}^k \gamma_j e^{c_j t}) + (\Omega^2 + \sum_{j=1}^k \gamma_j \Omega_j)(\sum_{j=0}^k \gamma_j e^{c_j t}) + (\sum_{j=0}^k \gamma_j e^{c_j t})(c_j^2 \sum_{j=0}^k \gamma_j e^{c_j t})^2 = 0. \tag{30}$$

For $v = \sum_{j=0}^k \gamma_j e^{c_j t} \neq 0$, this equation is simplified in form

$$c_j^2 = -\frac{\Omega^2 + \sum_{j=1}^k \gamma_j \Omega_j}{1 + c_j^2 (\sum_{j=0}^k \gamma_j e^{c_j t})^2}. \tag{31}$$

Here, we accept that the solution of equation (28) implicate v_0 as a leading term. Hence, the equation provided by v_0 should be in the form:

$$v_0'' + f(\alpha)v_0 = 0. \tag{32}$$

In this case, characteristic equation becomes as

$$c_0^2 = -f(\alpha). \tag{33}$$

Using (31), (33) and $v_0 = \gamma_0 e^{c_0 t} = \alpha e^{c_0 t}$, we can write

$$c_0^2 = \frac{\Omega^2}{1+\alpha} = f(\alpha). \tag{34}$$

Here, $f(\alpha)$ is an integral number. In equation (34), the coefficient in the denominator is neglected so that the calculations are not complicated. In view of the (29) and $f(\alpha)=\frac{\Omega^2}{1+\alpha}$, we have:

$$v''+\frac{\Omega^2}{1+\alpha}v-\frac{\Omega^2}{1+\alpha}v+v(v'')^2=0. \tag{35}$$

We construct the following homotopy and consider the linear term as $v''+\frac{\Omega^2}{1+\alpha}v$

$$v''+\frac{\Omega^2}{1+\alpha}v+\rho[v(v'')^2]=0. \tag{36}$$

Substituting Eqs. (29) and initial conditions (25) into homotopy (36) and equating the terms with identical powers of ρ, we obtain the following set of linear differential equations:

$$\rho^0: v_0''+(\Omega^2+\frac{\Omega^2}{1+\alpha})v_0=0,\ v_0(0)=\alpha,\quad v_0'(0)=0 \tag{37}$$

$$\rho^1: v_1''+(\Omega^2+\tfrac{\Omega^2}{1+\alpha})v_1+[\Omega_1+(v_0'')^2]v_0=0,\quad v_1(0)=0,\quad v_1'(0)=0. \tag{38}$$

Solving Eq. (37), we get

$$v_0=\alpha cos(\Omega\sqrt{\frac{2+\alpha}{1+\alpha}}t). \tag{39}$$

If the first-order approximation is enough, then setting $\rho=1$, we have

$$\Omega^2+\Omega_1=0. \tag{40}$$

Substituting (39) and (40) into Eq. (38) yields

$$v_1''+\Omega^2v_1+\tfrac{\Omega^2}{1+\alpha}v_1+\alpha\Omega^2[-1+\tfrac{3\alpha^2\Omega^2}{4}(\tfrac{2+\alpha}{1+\alpha})^2]cos(\Omega\sqrt{\tfrac{2+\alpha}{1+\alpha}}t)+\tfrac{\alpha^3\Omega^4}{4}(\tfrac{2+\alpha}{1+\alpha})^2cos(3\Omega\sqrt{\tfrac{2+\alpha}{1+\alpha}}t)=0. \tag{41}$$

Eliminating the secular term, we have

$$-1+\frac{3\alpha^2\Omega^2}{4}(\frac{2+\alpha}{1+\alpha})^2=0.$$

Hence, from the above equation, we can obtain

$$\Omega=\frac{2}{\sqrt{3}\alpha}(\frac{1+\alpha}{2+\alpha}). \tag{42}$$

By substituting Eq. (42) in Eq. (41), the solution v_1 is found

$$v_1=\frac{\alpha^3\Omega^2}{32}(\frac{2+\alpha}{1+\alpha})[cos(3\Omega\sqrt{\frac{2+\alpha}{1+\alpha}}t)-cos(\Omega\sqrt{\frac{2+\alpha}{1+\alpha}}t)]. \tag{43}$$

Thus, we obtain the first-order approximation by setting $\rho=1$:

$$v=v_0+v_1=\alpha cos(\Omega\sqrt{\tfrac{2+\alpha}{1+\alpha}}t)+\tfrac{\alpha^3\Omega^2}{32}(\tfrac{2+\alpha}{1+\alpha})[cos(3\Omega\sqrt{\tfrac{2+\alpha}{1+\alpha}}t)-cos(\Omega\sqrt{\tfrac{2+\alpha}{1+\alpha}}t)]. \tag{44}$$

The first-order approximate solution of HPM of the problem may be found as

$$v=\alpha cos(\Omega t)+\frac{\alpha^3\Omega^2}{32}[cos(3\Omega t)-cos(\Omega t)]. \tag{45}$$

As Ex. 1, here is seen a similar behavior the comparisons of the solutions for distinct t and α values. It is, again, noticable that in HPM, the difference between zero-order and 1st order solutions diverges when t gets larger. But, zero-order and 1st order solutions for MHPM are consistent even for large t values. Therefore, in this example, MHPM works better comparison to HPM for least order solutions. It can be seen that similar results can be obtained by solving this example with He's frequency amplitude formulation, which originates from an ancient Chinese mathematician (Geng & Cai, 2007; He, 2021a).

4 Conclusions

In this paper, we applied the modified form of HPM to two examples to show the effectiviness of the modification. As it is noticeable from the Figs. 1., 2, 3 and 4. modified solution represent the natural behaviour of the solution. Because, the arguments of cos and sine functions in solution (23) and (24) involves parameter α as well as Ω which smooths the solution. But classical homotopy does not involve parameter α but involves only Ω which is a drawback for smoothing the solution. Therefore, modified solution represent the behaviour of the problem better comparing to classical homotopy method. Ex. 2. shows a similar behavior. As a result the modified form of HPM appears to proved a constructive modification to nonlinear oscillators. This modification can be applied to many other nonlinear oscillators and partial differential equations.

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